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Investigating Elastic Stability of Cylindrical Shell with an Elastic Core Under Axial Compression by Energy Method

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Abstract

In this paper, the elastic axisymmetric buckling of a thin, isotropic and simply supported cylindrical shell with an elastic core under axial compression has been analyzed using energy method. The nonlinear strain-displacement relations in general cylindrical coordinates are simplified using Sanders kinematic relations (Sanders, 1963) for axial compression. Equilibrium equations are obtained by using minimum potential energy together with Euler equations applied for potential energy function in cylindrical shell. To acquire stability equation of cylindrical shell with an elastic core, minimum potential energy theory and Trefftz criteria are implemented. Stability and compatibility equations for an imperfect cylindrical shell with an elastic core are also obtained by the energy method, and the buckling analysis of shell is carried out using Galerkin method. Critical load curves versus the aspect ratio are obtained and analyzed for a cylindrical shell with an elastic core. It is concluded that the application of an elastic core increases elastic stability and significantly reduces the weight of cylindrical shells.

Keywords: Axial compression; Cylindrical shells; Elastic core; Elastic buckling; Critical Load; Imperfection; Galerkin method

1. Introduction

Thin walled cylindrical shells are widely used in a variety of industries including rockets, missiles, aircraft fuselages, submarines, silos, pressure vessels etc. Cylindrical shells in engineering structures with large aspect ratios are typically stiffened against buckling by circumferential and longitudinal members, known as ring and stringers stiffeners, respectively. Accordingly, sandwich construction with lightweight core materials has well known advantages of stiffness and strength to weight. These structures are widely used in applications where weight consideration has great importance.

Cylindrical shell with an elastic core under axial pressure is well known to be highly efficient in terms of combining high stiffness to weight. These structures consist of a fully condensed outer shell material which is supported by a low density cellular core.

Karam et al. (1995) analyzed the elastic buckling of a thin cylindrical shell supported by an elastic core. The results they reported show that this structural configuration achieves significant weight saving compared with a hollow cylinder.

Agrawal et al. (1997) investigated the weight compressions of optimized stiffened, unstiffened, and sandwich cylindrical shells. They determined the weight advantage of honeycomb sandwich cylindrical shells under axial compression compared with axially stiffened cylinders. By comparing weight optimized shells, they showed that honeycomb sandwiches offer a substantial weight advantage over a significant load range.

Hutchinson et al. (2000) studied buckling of

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cylindrical shells with metal foam cores subjected to axial compression. They obtained optimal outer shell thickness, core thickness and core density by minimizing the weight of geometrically perfect shell with a specified load carrying capacity. The results reported suggest that for sandwich shells and curved sandwich panels there is a practically significant loading range for which optimized sandwich constructions with metal foam cores, have a distinct weight advantage over metal stringer constructions. However, these results also indicate that the imperfection sensitivity is essentially as severe as it is for the monocoque cylinder.

In this research, the elastic buckling of a cylindrical shell with an elastic core has been analyzed using the energy method, and its buckling resistance was compared with that of a hollow cylindrical shell with equal mass. One of the main objectives of this research is to find the effect of an elastic core on the reduction of a cylindrical shell weight.

The buckling of a simply supported imperfect core filled cylindrical shell under axial compression is also considered here. The effect of initial imperfection on the linear elastic stability of a cylindrical shell with an elastic core is analyzed by using the Galerkin method.

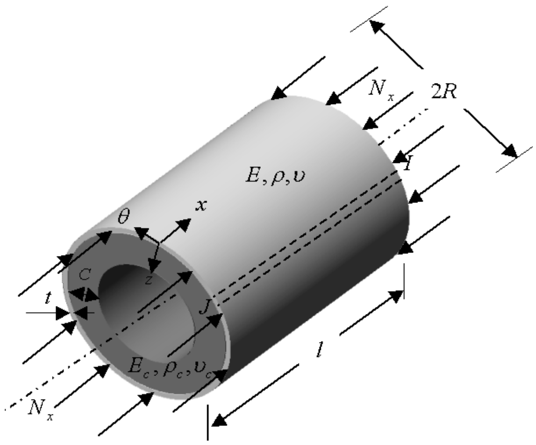


Fig. 1. Geometry of the cylindrical shell with a core and loading configuration.

2. Analysis

An isotropic, thin-walled, closed-ended, perfectly circular cylindrical shell with an elastic core with simply supported end conditions has been considered in this research. The shell has a uniform thickness t, radius R, length l, density ρ, Young's modulus E and Poisson ratio υ. The elastic core properties are: density ρ_c, Young's modulus E_c, Poisson ratio υ_c, and a uniform thickness C as shown in Fig. 1. Coordinates x, θ, and z axes are defined with corresponding displacements u, v and w.

The normal and shear strains at a distance z from the middle planes of the shell are (Donnel, 1976):

$$\begin{aligned} \varepsilon_x &= \varepsilon_{xm} + zk_x \\ \varepsilon_\theta &= \varepsilon_{\theta m} + zk_\theta \\ \gamma_{x\theta} &= \gamma_{x\theta m} + zk_{x\theta} \end{aligned} \quad (1)$$

Where ε's are the normal strains, γ's are the shear strains, and k_{ij} are the curvatures. The subscript m refers to the strain at the middle surface of shell. The indices x and θ refer to the axial and circumferential directions, respectively. According to the Sanders kinematic relations (Sanders, 1963), the general strain-displacement relations can be simplified to give the following equations for the strains at the middle surface and the curvatures in terms of displacement components:

$$\begin{aligned} \varepsilon_{xm} &= u_{,x} + 0.5w^2{}_{,x} \\ \varepsilon_{\theta m} &= (v_{,\theta} + w)/R + (v - w_{,\theta})^2/2R^2 \\ \gamma_{x\theta m} &= u_{,\theta}/R + v_{,x} + (-w_{,x}v + w_{,x}w_{,\theta})/R \\ k_x &= -w_{,xx} \\ k_\theta &= (v_{,\theta} - w_{,\theta\theta})/R^2 \\ k_{x\theta} &= (v_{,x} - 2w_{,x\theta})/2R \end{aligned} \quad (2)$$

Where u, v and w are the displacements and (,) indicates a partial derivative. Hook's law in terms of forces and moments per unit length is:

$$\begin{aligned} N_x &= C(\varepsilon_{xm} + \upsilon\varepsilon_{\theta m}) \\ N_\theta &= C(\varepsilon_{\theta m} + \upsilon\varepsilon_{xm}) \\ N_{x\theta} &= N_{\theta x} = C(1-\upsilon)\gamma_{x\theta m}/2 \\ M_x &= D(k_x + \upsilon k_\theta) \\ M_\theta &= D(k_\theta + \upsilon k_x) \\ M_{x\theta} &= M_{\theta x} = D(1-\upsilon)k_{x\theta} \end{aligned} \quad (3)$$

Where N_{ij} and M_{ij} are related to σ_{ij} through the shell thickness according to the first-order shell theory, E is the elastic modulus, υ is the Poisson ratio, D is the flexural rigidity of the shell ($D = Et^3/12(1-\upsilon^2)$), and $C = Et/(1-\upsilon^2)$.

3. Energy formulation

The total potential energy of the shell with an elastic core under axial compressive force is the sum of membrane strain energy U_m, the bending strain energy U_b, the strain energy stored in the core U_e, and the potential energy Ω of the applied compressive force expressed as:

$$V = U_m + U_b + U_e + \Omega \tag{4}$$

Where:

$$\begin{aligned} U_m &= RC/2 \iint [\varepsilon^2{}_{xm} + \varepsilon^2{}_{\theta m} + 2\upsilon \varepsilon_{xm}\varepsilon_{\theta m} \\ &\quad + (1-\upsilon)\gamma^2{}_{x\theta m}/2] dx d\theta \\ U_b &= RD/2 \iint [K^2{}_x + K^2{}_\theta + 2\upsilon K_x K_\theta \\ &\quad + 2(1-\upsilon)K^2{}_{x\theta}] dx d\theta \\ U_e &= \frac{k_e}{2} \int_x \int_\theta w^2 R dx d\theta \\ \Omega &= -\iint (P_x u) R\, dx\, d\theta \end{aligned} \tag{5}$$

In which k_e is the spring constant for the compliant core. For zero spring stiffness this reduces to the result for a hollow cylindrical shell. Assuming that the shell thickness is much smaller that its radius, the spring constant k_e can be found from the result of stress in the z direction of a flat strip element (IJ in Fig. 1) with an elastic foundation subjected to a sinusoidal displacement in the z direction (Gough et al., 1940; Allen, 1969):

$$\sigma_z = -\frac{2\pi E_c}{(3-\upsilon_c)(1+\upsilon_c)} \cdot \frac{m}{l} w \tag{6}$$

Where σ_z, m and l are stress in the z direction, longitudinal wave number and shell length, respectively. For a cylindrical shell with a core under internal pressure q, we have

$$q = \sigma_z = -k_e w \tag{7}$$

Where w is the radial displacement of the core. Equations (6) and (7) lead to:

$$k_e = \frac{2E_c}{(3-\upsilon_c)(1+\upsilon_c)} \cdot \frac{1}{\lambda} \tag{8}$$

Where λ is buckling wavelength parameter ($\lambda = \frac{l}{m\pi}$).

Assuming that the cylindrical shell with an elastic core is under axial load alone, the total potential energy is a function of displacement components and their derivatives and can be written as:

$$\begin{aligned} U = \iiint F(&u, v, w, u_{,x}, u_{,\theta}, v_{,x}, v_{,\theta}, w_{,x}, \\ &w_{,\theta}, w_{,xx}, w_{,\theta\theta}, w_{,x\theta}) dx d\theta dz \end{aligned} \tag{9}$$

Minimizing the potential energy function leads to the Euler equations:

$$\begin{aligned} &\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\frac{\partial F}{\partial u_{,x}} - \frac{\partial}{\partial \theta}\frac{\partial F}{\partial u_{,\theta}} = 0 \\ &\frac{\partial F}{\partial v} - \frac{\partial}{\partial x}\frac{\partial F}{\partial v_{,x}} - \frac{\partial}{\partial \theta}\frac{\partial F}{\partial v_{,\theta}} = 0 \\ &\frac{\partial F}{\partial w} - \frac{\partial}{\partial x}\frac{\partial F}{\partial w_{,x}} - \frac{\partial}{\partial \theta}\frac{\partial F}{\partial w_{,\theta}} + \frac{\partial^2}{\partial x^2}\frac{\partial F}{\partial w_{,xx}} \\ &\quad + \frac{\partial^2}{\partial x \partial \theta}\frac{\partial F}{\partial w_{,x\theta}} + \frac{\partial^2}{\partial \theta^2}\frac{\partial F}{\partial w_{,\theta\theta}} = 0 \end{aligned} \tag{10}$$

Upon substitution from Eq. (9) into Eq. (10) and using Eqs. (2)-(5), the equilibrium equations for general thin cylindrical shell with a core are obtained as:

$$\begin{aligned} &RN_{x,x} + N_{x\theta,\theta} = -PR \\ &N_\theta \beta_\theta + N_{x\theta}\beta_x - RN_{x\theta,x} - M_{x\theta,x} \\ &\quad - N_{\theta,\theta} - \frac{1}{R} M_{\theta,\theta} = 0 \\ &RM_{x,xx} + 2M_{x\theta,x\theta} + \frac{1}{R} M_{\theta,\theta\theta} - N_\theta - \left[RN_x \beta_{x,x} + \right. \\ &\left. N_{x\theta}\left(R\beta_{\theta,x} + \beta_{x,\theta}\right) + N_\theta \beta_{\theta,\theta} \right] \\ &\quad + \left(RN_{x\theta,x} + N_{\theta,\theta}\right)\beta_\theta + k_e R w = 0 \end{aligned} \tag{11}$$

Where

$$\beta_x = w_{,x} \quad \text{and} \quad \beta_\theta = (v - w_{,\theta})/R \tag{12}$$

are the rotation in the x and θ directions. Recall that the transverse shear force in the circumferential direction is:

$$Q_\theta = M_{\theta,\theta}/R + M_{x\theta,x} \tag{13}$$

This shows that in the Donnel equations for short cylindrical shells the shear force in the circumferential direction Q_θ, rotations β_x and β_θ are ignored (Brush et al., 1975). Therefore, Eq. (11) can be written as:

$$
\begin{aligned}
&RN_{x,x} + N_{x\theta,\theta} = -PR \\
&RN_{x\theta,x} + N_{\theta,\theta} = 0 \\
&RM_{x,xx} + 2M_{x\theta,x\theta} + \frac{1}{R} M_{\theta,\theta\theta} - N_\theta - RN_x \beta_{x,x} - \\
&N_{x\theta}\left(R\beta_{\theta,x} + \beta_{x,\theta}\right) - N_\theta \beta_{\theta,\theta} + k_e R w = 0
\end{aligned} \tag{14}
$$

4. Minimum potential energy criterion

In this section the stability equations can be derived on the basis of the minimum potential energy criterion. If V is the total potential energy of the shell, its variation in equilibrium state using the Taylor series leads to:

$$\Delta V = \delta V + \frac{1}{2!}\delta^2 V + \frac{1}{3!}\delta^3 V + \ldots \tag{15}$$

The first term of variation δV is associated with the state of equilibrium. The stability of the original configuration of the shell in the neighborhood of equilibrium state can be determined by the sign of the second variation $\delta^2 V$ as follows:

The equilibrium is stable if $\delta^2 V > 0$ for all virtual displacements.

The equilibrium is unstable if $\delta^2 V < 0$ for at least one admissible set of virtual displacements.

The condition $\delta^2 V = 0$ is used to derive the stability equations for many buckling problems as discussed by Langhaar(1962).

If the state of stable equilibrium of a general cylindrical shell with a core under axial compressive load P is designated by u_0, v_0 and w_0, the displacement components of the neighboring state are:

$$
\begin{aligned}
&u \to u_0 + u_1 \\
&v \to v_0 + v_1 \\
&w \to w_0 + w_1
\end{aligned} \tag{16}
$$

Similarly, the components of forces and moments related to the neighboring state are related to the state of equilibrium according to the following relations:

$$
\begin{aligned}
&N_x \to N_{x0} + \Delta N_x \quad && M_x \to M_{x0} + \Delta M_x \\
&N_\theta \to N_{\theta 0} + \Delta N_\theta \quad && M_\theta \to M_{\theta 0} + \Delta M_\theta \\
&N_{x\theta} \to N_{x\theta 0} + \Delta N_{x\theta} \quad && M_{x\theta} \to M_{x\theta 0} + \Delta M_{x\theta}
\end{aligned} \tag{17}
$$

If $N_{x1}, N_{\theta 1}, N_{x\theta 1}, \ldots$ express the linear portion of $\Delta N_x, \Delta N_\theta, \Delta N_{x\theta}, \ldots$ in terms of u_1, v_1 and w_1 then:

$$
\begin{aligned}
&N_{x1} = C(\varepsilon_{x1} + \upsilon \varepsilon_{\theta 1}) \\
&N_{\theta 1} = C(\varepsilon_{\theta 1} + \upsilon \varepsilon_{x1}) \\
&N_{x\theta 1} = N_{\theta x1} = C(\frac{1-\upsilon}{2})\gamma_{x\theta 1} \\
&M_{x1} = D(K_{x1} + \upsilon K_{\theta 1}) \\
&M_{\theta 1} = D(K_{\theta 1} + \upsilon K_{x1}) \\
&M_{x\theta 1} = M_{\theta x1} = D(1-\upsilon)K_{x\theta 1}
\end{aligned} \tag{18}
$$

and

$$
\begin{aligned}
&N_{x0} = C(\varepsilon_{x0} + \upsilon \varepsilon_{\theta 0}) \\
&N_{\theta 0} = C(\varepsilon_{\theta 0} + \upsilon \varepsilon_{x0}) \\
&N_{x\theta 0} = C(\frac{1-\upsilon}{2})\gamma_{x\theta 0} \\
&M_{x0} = D(K_{x0} + \upsilon K_{\theta 0}) \\
&M_{\theta 0} = D(K_{\theta 0} + \upsilon K_{x0}) \\
&M_{x\theta 0} = D(1-\upsilon)K_{x\theta 0}
\end{aligned} \tag{19}
$$

Using Eq. (16) for the displacement components of a neighboring state of stable equilibrium and employing the decomposed linear part of strain displacement relations, the linear strain relations for the equilibrium state and its first variation, subscripted by (0) and (1) respectively, are obtained as:

$$
\begin{aligned}
&e_{xx} \to e_{xx0} + e_{xx1} \quad K_x \to K_{x0} + K_{x1} \\
&e_{\theta\theta} \to e_{\theta\theta 0} + e_{\theta\theta 1} \quad K_\theta \to K_{\theta 0} + K_{\theta 1} \\
&e_{x\theta} \to e_{x\theta 0} + e_{x\theta 1} \quad K_{x\theta} \to K_{x\theta 0} + K_{x\theta 1} \\
&\beta_x \to \beta_{x0} + \beta_{x1} \\
&\beta_\theta \to \beta_{\theta 0} + \beta_{\theta 1}
\end{aligned} \tag{20}
$$

Substituting into the potential energy function yields:

$$
\begin{aligned}
V_0 + \Delta V = {} & \frac{RC}{2} \int_\theta \int_x \{[(e_{xx0} + e_{xx1}) + \frac{1}{2}(\beta_{x0} + \beta_{x1})^2]^2 \\
& + [(e_{\theta\theta 0} + e_{\theta\theta 1}) + \frac{1}{2}(\beta_{\theta 0} + \beta_{\theta 1})^2]^2 \\
& + 2\upsilon[(e_{xx0} + e_{xx1}) + \frac{1}{2}(\beta_{x0} + \beta_{x1})^2][(e_{\theta\theta 0} + e_{\theta\theta 1}) \\
& + \frac{1}{2}(\beta_{\theta 0} + \beta_{\theta 1})^2] + \frac{1-\upsilon}{2}[(e_{x\theta 0} + e_{x\theta 1}) \\
& + (\beta_{x0} + \beta_{x1})(\beta_{\theta 0} + \beta_{\theta 1})]^2\} dx d\theta \\
& + \frac{RD}{2} \int_\theta \int_x [(K_{x0} + K_{x1})^2 + (K_{\theta 0} + K_{\theta 1})^2 \\
& + 2(1-\upsilon)(K_{x\theta 0} + K_{x\theta 1})^2] dx d\theta
\end{aligned} \tag{21}
$$

$$+\frac{k_e R}{2}\int_\theta\int_x (w_0+w_1)^2\,dx\,d\theta-\int_\theta\int_x [P(u_0+u_1)]R\,dx\,d\theta$$

Terms with subscript "$_0$" and "$_1$" are related to V_0 and ΔV, respectively. Neglecting the pre-buckling rotations β_{x0} and $\beta_{\theta 0}$, the second variation of the potential energy is obtained as:

$$\begin{aligned}\frac{1}{2}\delta^2 V=&\frac{RC}{2}\int_\theta\int_x (e_{xx1}^2+e_{\theta\theta 1}^2+2\nu e_{xx1}e_{\theta\theta 1}\\&+\frac{1-\upsilon}{2}e_{x\theta 1}^2)dxd\theta\\&+\frac{R}{2}\int_\theta\int_x (N_{x0}\beta_{x1}^2+N_{\theta 0}\beta_{\theta 1}^2+2N_{x\theta 0}\beta_{x1}\beta_{\theta 1})dxd\theta\\&+\frac{RD}{2}\int_\theta\int_x (K_{x1}^2+K_{\theta 1}^2+2\upsilon K_{x1}K_{\theta 1}+2(1-\upsilon)K_{x\theta 1}^2)dxd\theta\\&+\frac{k_e R}{2}\int_x\int_\theta w_1{}^2 dxd\theta\end{aligned}\qquad(22)$$

Where the pre-buckling linearized forces are given in Eq. (19). Substituting for the strains and curvatures from the linearized strain-displacement relations yields:

$$\begin{aligned}&\varepsilon_{x1}=e_{xx1}=u_{1,x} && K_{x1}=-w_{1,xx}\\&\varepsilon_{\theta 1}=e_{\theta\theta 1}=\frac{v_{1,\theta}+w_1}{R} && K_{\theta 1}=\frac{v_{1,\theta}-w_{1,\theta\theta}}{R^2}\\&\gamma_{x\theta 1}=e_{x\theta 1}=v_{1,x}+\frac{u_{1,\theta}}{R} && K_{x\theta 1}=\frac{1}{2}.\frac{v_{1,x}-2w_{1,x\theta}}{R}\\&\beta_{x1}=-w_{1,xx} && \beta_{\theta 1}=\frac{v_1-w_{1,\theta}}{R}\end{aligned}\qquad(23)$$

5. Stability equations

Substitution of Eq. (22) into the Euler Eq. (10) leads to the stability equations:

$$\begin{aligned}&RN_{x1,x}+N_{x\theta 1,\theta}=0\\&RN_{x\theta 1,x}+N_{\theta 1,\theta}+\frac{1}{R}M_{\theta 1,\theta}+M_{x\theta 1,x}\\&\quad-(N_{\theta 0}\beta_{\theta 1}+N_{x\theta 0}\beta_{x1})=0\\&RM_{x1,xx}+2M_{x\theta 1,x\theta}+\frac{1}{R}M_{\theta 1,\theta\theta}-N_{\theta 1}\\&\quad-[RN_{x0}\beta_{x1,x}+N_{x\theta\ 0}(R\beta_{\theta 1,x}+\beta_{x1,\theta})\\&\quad+N_{\theta\ 0}\beta_{\theta 1,\theta}]+(RN_{x\theta\ 0,x}+N_{\theta\ 0,\theta})\beta_{\theta 1}\\&\quad+k_e Rw_1=0\end{aligned}\qquad(24)$$

In Eq. (24), the transverse shear force Q_θ and rotations β_x and β_θ are ignored and the stability equations reduce to the Donnel equations.

$$\begin{aligned}&RN_{x1,x}+N_{x\theta 1,\theta}=0\\&RN_{x\theta 1,x}+N_{\theta 1,\theta}=0\\&RM_{x1,xx}+2M_{x\theta 1,x\theta}+\frac{1}{R}M_{\theta 1,\theta\theta}-N_{\theta 1}-[RN_{x0}\beta_{x1,x}\\&+N_{x\theta 0}(R\beta_{\theta 1,x}+\beta_{x1,\theta})+N_{\theta 0}\beta_{\theta 1,\theta}]+k_e Rw_1=0\end{aligned}\qquad(25)$$

By substituting N_{ij} and M_{ij} with their strain equivalences from Eq. (18), the Donnel equations are obtained in terms of displacements from Eq. (23) as:

$$\begin{aligned}&Ru_{1,xx}+\frac{1-\upsilon}{2R}u_{1,\theta\theta}+\frac{1+\upsilon}{2}v_{1,x\theta}+\nu w_{1,x}=0\\&\frac{CR^2+D}{R^3}v_{1,\theta\theta}+\frac{(1-\upsilon)(D+R^2C)}{2R}v_{1,xx}\\&\qquad+\frac{C(1+\upsilon)}{2}u_{1,x\theta}\\&-\frac{D}{R^3}w_{1,\theta\theta\theta}-\frac{D}{R}w_{1,xx\theta}+(\frac{C+N_{\theta 0}}{R})w_{1,\theta}\\&-\frac{v_1}{R}N_{\theta 0}=0\\&Dw_{1,xxxx}+\frac{2D}{R^2}w_{1,xx\theta\theta}+\frac{Dw_{1,\theta\theta\theta\theta}}{R^4}\\&\quad-\frac{D}{R^2}(v_{1,xx\theta}+\frac{1}{R^2}v_{1,\theta\theta\theta})\\&\quad+\frac{C}{R^2}(v_{1,\theta}+w_1+\upsilon Ru_{1,x})-N_{x0}.w_{1,xx}\\&\quad-\frac{N_{\theta 0}}{R^2}w_{1,\theta\theta}+\frac{N_{\theta 0}}{R^2}v_{1,\theta}+k_e w_1=0\end{aligned}\qquad(26)$$

6. Axial compression

A simply supported cylinder with an elastic core subjected to a uniformly distributed axial compressive load P is considered. The initial deformation is axisymmetric, and the axial compressive buckling load of shell with an elastic core p_{cr}, is the lowest load at which equilibrium in the axisymmetric form ceases to be stable. From a membrane analysis of the unbuckled cylindrical form (Brush et al., 1975), the prebuckling forces are

$$N_{x0}=-\frac{P}{2\pi R}\quad,\quad N_{x\theta 0}=N_{\theta 0}=0\qquad(27)$$

Equations (26) are the stability equations in coupled form with the variables u_1, v_1, w_1 and may be partially uncoupled and written in the following form (Brush et al., 1975):

$$\nabla^4 u_1 = -\frac{\upsilon}{R} w_{1,xxx} + \frac{1}{R^3} w_{1,x\theta\theta}$$
$$\nabla^4 v_1 = -\frac{1}{R^4} w_{1,\theta\theta\theta} - \frac{2+\upsilon}{R^2} w_{1,xx\theta}$$
$$D\nabla^8 w_1 + (\frac{1-\upsilon^2}{R^2})C.w_{1,xxxx} - \nabla^4[N_{x0}w_{1,xx} + \frac{2}{R}N_{x\theta 0}w_{1,x\theta} + \frac{N_{\theta 0}w_{1,\theta\theta}}{R^2} - k_e w_1] = 0 \quad (28)$$

Where $\nabla^4 = (\frac{\partial^4}{\partial x^4} + \frac{2}{R^2}.\frac{\partial^4}{\partial x^2 \partial\theta^2} + \frac{1}{R^4}.\frac{\partial^4}{\partial\theta^4})$. Substituting prebuckling forces from Eq. (27) into the third Eq. (28) yields:

$$D\nabla^8 w_1 + \left(\frac{1-\upsilon^2}{R^2}\right)Cw_{1,xxxx} + \nabla^4\left[\frac{P_x w_{1,xx}}{2\pi R} + k_e w_1\right] = 0 \quad (29)$$

Where $\nabla^8 w_1 = \nabla^4(\nabla^4 w_1)$. Equation (29) has constant-coefficients and from the edge conditions:

$$w_1 = w_{1,xx} = 0 \quad (30)$$

Assume a solution as:

$$w_1 = C_1 \sin \overline{m}x \sin n\theta \quad (31)$$

Where $\overline{m} = m\pi R/L$; m and n are the axial half-wave number and the circumferential full-wave number , respectively, and C_1 is the amplitude of the displacement function.

Substituting the solution w_1 from Eq. (31) into Eq. (29) results in:

$$D\left(\overline{m}^2 + \frac{n^2}{R^2}\right)^4 w_1 + \frac{1-\upsilon^2}{R^2}C\overline{m}^4 w_1 - \frac{P}{2\pi R}\overline{m}^2\left(\overline{m}^2 + \frac{n^2}{R^2}\right)^2 w_1 + k_e\left(\overline{m}^2 + \frac{n^2}{R^2}\right)^2 w_1 = 0 \quad (32)$$

Dividing Eq. (32) by $\left(\overline{m}^2 + \frac{n^2}{R^2}\right)^2 .w$ gives:

$$D\left(\overline{m}^2 + \frac{n^2}{R^2}\right)^2 + \frac{(1-\upsilon^2)}{R^2}C\frac{\overline{m}^4}{\left(\overline{m}^2 + \frac{n^2}{R^2}\right)^2} - \frac{P}{2\pi R}\overline{m}^2 + k_e = 0 \quad (33)$$

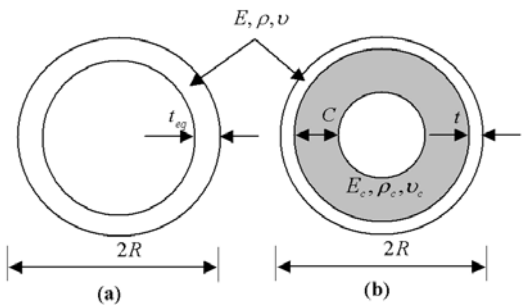


Fig. 2. (a) Thin walled cylindrical shell without a core. (b) Thin walled cylindrical shell with a core of depth C of an equal radius and mass as shell in (2a).

Rearranging Eq. (33) leads to:

$$\frac{P}{2\pi R} = \frac{D\left(\overline{m}^2 + \frac{n^2}{R^2}\right)^2 + \frac{(1-\upsilon^2)}{R^2}C\frac{\overline{m}^4}{\left(\overline{m}^2 + \frac{n^2}{R^2}\right)^2} + k_e}{\overline{m}^2} \quad (34)$$

In Eq. (34), the minimum axial load is the critical axial load p_{cr} , which can be obtained by specifying values for m and n .

The performance of a thin walled cylindrical shell with a compliant elastic core can be evaluated by comparing its buckling resistance with an empty shell having equal diameter and mass (Fig. 2). For calculating the equivalent thickness t_{eq} of an empty shell with a shell with an elastic core and equal mass and radius, the following equation can be written:

$$\rho \times 2\pi R t_{eq} = \rho \times 2\pi R t + \rho_c \pi (R^2 - a^2) \quad (35)$$

Therefore:

$$t_{eq} = t[1 + \frac{c}{2t}\frac{\rho_c}{\rho}(2 - \frac{c}{R})] \quad (36)$$

7. Cylindrical shell with axisymmetric imperfection

The equilibrium equations can be simplified substantially by using polar coordinate s defined by the relation:

$$ds = Rd\theta \quad (37)$$

In terms of this coordinate S , Eq. (14) may be rewritten as follows: (Brush et al., 1975)

$$\begin{aligned} &N_{x,x} + N_{xs,s} = -P \\ &N_{xs,x} + N_{s,s} = 0 \\ &D\nabla^4 w + \frac{1}{R} N_s - (N_x w_{,xx} + 2N_{xs} w_{,xs} + N_s w_{,ss}) \\ &+k_e w = 0 \end{aligned} \tag{38}$$

For a slightly imperfect shell, let $w^*(x)$ denote a known small imperfection, i.e., a small deviation of the shell middle surface from a circular cylindrical shape, given by (Brush et al., 1975):

$$w_{total} = w + w^*(x) \tag{39}$$

By substitution of Eq. (39) into the third Eq. (38), the equilibrium equation in the radial direction can be written as:

$$\begin{aligned} &D\nabla^4 w + \frac{1}{R} N_s - [N_x (w + w^*)_{,xx} \\ &\quad + 2N_{xs} w_{,xs} + N_s w_{,ss}] + k_e w = 0 \end{aligned} \tag{40}$$

8. Pre–buckling analysis

Since the shell with an elastic core and the applied axial compressive force are both axi-symmetric, shell configurations on the primary equilibrium path are axisymmetric as well. For the axisymmetric configurations on the primary path $w_0 = w_0(x)$, $v_0 = 0$, $u_0 = u_0(x)$ (Brush et al., 1975), for the axisymmetric loading, $N_{xs0} = 0$, and from the first equilibrium equation N_{xx0} is seen to be independent of x . Considering cylinder both ends conditions:

$$N_{x0} = -\frac{P}{2\pi R} \tag{41}$$

By applying these expressions into the equilibrium equations and simplifying in accordance with the procedure explained before then:

$$\begin{aligned} &D\nabla^4 w_0 + \frac{1}{R} N_{s0} - [-\frac{P}{2\pi R}(w_0 + w^*)_{,xx} \\ &\quad + 2N_{xs0} w_{0,xs} + N_{s0} w_{0,ss}] + k_e w_0 = 0 \end{aligned} \tag{42}$$

Where $w_0 = w_0(x)$ and:

$$w_{0,xs} = w_{0,ss} = 0 \tag{43}$$

Substitution of these expressions into Eq. (42) yields:

$$D\frac{d^4 w}{dx^4} + \frac{1}{R} N_{s0} + \frac{P}{2\pi R} \frac{d^2 (w_0 + w^*)}{dx^2} + k_e w_0 = 0 \tag{44}$$

Where R is undeformed-middle-surface radius of the cylindrical shell. Then specializing the constitutive and kinematic relations for axial symmetry leads to:

$$N_{s0} = Eh\varepsilon_{s0} + \upsilon N_{x0} \tag{45}$$

Where

$$\varepsilon_{s0} = \frac{w_0}{R} \tag{46}$$

By substitution of Eqs. (45) and (46) into Eq. (44) and rearrangement, the prebuckling equilibrium equa-tion now may be written in the following form:

$$Dw_0^{iv} + \frac{P}{2\pi R} w_0^{''} + \left(\frac{Et}{R^2} + k_e\right) w_0 = \frac{P}{2\pi R}\left(\frac{\upsilon}{R} - w^{*''}\right) \tag{47}$$

The Koiter (1963) model for the axisymmetric geometrical imperfection of cylindrical shell is:

$$w^*(x) = -\mu t \cos(\frac{m\pi x}{l}) \quad , \quad -\frac{l}{2} \le x \le +\frac{l}{2} \tag{48}$$

Where m is the number of half waves in x direction, and μt represents the amplitude of imperfection of the middle surface of the shell ($0 \le \mu \le 1$). The unloaded shell in the imperfection form, including w^*, is assumed to be stress free. The small angles of rotation $w_{,x}$ in the equations for an initially perfect cylinder are replaced by $(w + w^*)_{,x}$ (Brush et al., 1975). The minus sign in Eq. (48) signifies that the imperfect shell bulges inward at $x = 0$. Substituting $w^*(x)$ from Eq. (48) into Eq. (47) and rearrangement gives:

$$\begin{aligned} &Dw_0^{iv} + \frac{P}{2\pi R} w_0^{''} + \left(\frac{Et}{R^2} + k_e\right) w_0 \\ &= \frac{P}{2\pi R}\left(\frac{\upsilon}{R} - \mu t \frac{m^2\pi^2}{l^2} \cos\left(\frac{m\pi x}{l}\right)\right) \end{aligned} \tag{49}$$

A solution of the following form is suggested as:

$$w_0 = \frac{\upsilon P}{2\pi\left(Et + k_e R^2\right)} + At\cos\left(\frac{m\pi x}{l}\right) \tag{50}$$

Substituting w_0 from Eq. (50) into Eq. (49) and rearranging gives:

$$\begin{aligned}&\left[DAt\frac{m^4\pi^4}{l^4} - \frac{P}{2\pi R}At\frac{m^2\pi^2}{l^2} + \left(\frac{Et}{R^2}+k_e\right)At\right]\\ &\cos\left(\frac{m\pi x}{l}\right)\\ &+\left(\frac{Et}{R^2}+k_e\right)\frac{\upsilon P}{2\pi\left(Et+k_eR^2\right)} = \frac{\upsilon P}{2\pi R^2}\\ &-\frac{P}{2\pi R}\mu t\frac{m^2\pi^2}{l^2}\cos\left(\frac{m\pi x}{l}\right)\end{aligned} \tag{51}$$

The constant coefficient A in terms of μ is obtained from:

$$A = \frac{-\dfrac{P}{2\pi R}\mu t\dfrac{m^2\pi^2}{l^2}}{Dt\dfrac{m^4\pi^4}{l^4} - \dfrac{P}{2\pi R}t\dfrac{m^2\pi^2}{l^2}\left(\dfrac{Et}{R^2}+k_e\right)t} \tag{52}$$

With this solution for the axisymmetric form, prebuckling coefficients are found to be:

$$\begin{aligned}&N_{xs0} = 0 \quad , \quad N_{x0} = -\frac{P}{2\pi R}\\ &N_{s0} = \frac{Et^2}{R}A\cos\left(\frac{m\pi x}{l}\right) + \frac{Et}{R}\frac{\upsilon P}{2\pi\left(Et+k_eR^2\right)}\\ &\qquad -\frac{\upsilon P}{2\pi R}\\ &w^*(x) = -\mu t\cos\left(\frac{m\pi x}{l}\right)\\ &w_0(x) = \frac{\upsilon P}{2\pi\left(Et+k_eR^2\right)} + At\cos\left(\frac{m\pi x}{l}\right)\end{aligned} \tag{53}$$

9. Compatibility and stability equations

The net strains and components of forces for the imperfect cylindrical shell now become: (Brush et al., 1975)

$$\begin{aligned}&e_{x1} = u_{1,x} + w_{1,x}\frac{dw_0}{dx} + \frac{dw^*}{dx}w_{1,x}\\ &e_{s1} = v_{1,s} + \frac{1}{R}w_1\\ &e_{xs1} = v_{1,x} + u_{1,s} + w_{1,s}\frac{dw_0}{dx} + w_{1,s}\frac{dw^*}{dx}\end{aligned} \tag{54}$$

$$\begin{aligned}&N_{x1} = C(e_{x1} + \upsilon e_{s1})\\ &N_{s1} = C(e_{s1} + \upsilon e_{x1})\\ &N_{xs1} = N_{\theta x1} = C(\frac{1-\upsilon}{2}).e_{xs1}\end{aligned}$$

For the imperfect cylindrical shell with an elastic core, the compatibility equation is obtained from Eq. (54):

$$\begin{aligned}&e_{x1,ss} + e_{s1,xx} - e_{xs1,xs} = \frac{1}{R}w_{1,xx} - w_{0,xx}w_{1,ss}\\ &\quad - w^*_{,xx}w_{1,ss}\end{aligned} \tag{55}$$

Introducing the Airy stress function f :

$$N_{x1} = f_{,ss} \ , \ N_{s1} = f_{,xx} \ , \ N_{xs1} = -f_{,xs} \tag{56}$$

and defining $C = \dfrac{Et}{1-\upsilon^2}$, after rearrangement then:

$$\begin{aligned}&e_{x1} = \frac{1}{Et}(N_{x1} - \upsilon N_{s1})\\ &e_{s1} = \frac{1}{Et}(N_{s1} - \upsilon N_{x1})\\ &e_{xs1} = 2\frac{1+\upsilon}{Et}N_{xs1}\end{aligned} \tag{57}$$

Therefore, the compatibility equation in terms of the Airy stress function and the lateral displacement component w_1 is given as follows:

$$\nabla^4 f - Et(\frac{1}{R}w_{1,xx} - w_{1,ss}\frac{d^2w_0}{dx^2} - w_{1,ss}\frac{d^2w^*}{dx^2}) = 0 \tag{58}$$

In accordance with the procedure in earlier section, the stability equations are written as:

$$\begin{aligned}&N_{x1,x} + N_{xs1,s} = 0\\ &N_{xs1,x} + N_{s1,s} = 0\\ &D\nabla^4 w_1 + \frac{1}{R}N_{s1} - (N_{x0}w_{1,xx} + N_{s0}w_{1,ss}\\ &\quad + w_0 N_{x1} + w^{*\prime\prime}N_{x1}) + k_e w_1 = 0\end{aligned} \tag{59}$$

The first and second stability equations are automatically satisfied and introduction of Eq. (56) into the third stability in Eq. (59), reduces to

$$\begin{aligned}&D\nabla^4 w_1 + \frac{1}{R}f_{,xx} - (N_{x0}w_{1,xx} + N_{s0}w_{1,ss}\\ &\quad + f_{,ss}\frac{d^2w_0}{dx^2} + f_{,ss}\frac{d^2w^*}{dx^2}) + k_e w_1 = 0\end{aligned} \tag{60}$$

The following non-dimensional parameters are defined by expressions:

$$\begin{aligned} &N_{x0} \rightarrow Et\bar{N}_{x0} \ , \ N_{s0} \rightarrow Et\bar{N}_{s0} \ , \ f \rightarrow Et^3 \phi \\ &w^* \rightarrow t\, w^* \ , \ w_0 \rightarrow tw_0 \ , \ w_1 \rightarrow tw_1 \end{aligned} \tag{61}$$

Therefore, in non-dimensional form, the stability and compatibility equations are given as follows:

$$\begin{aligned} &C\nabla^4 w_1 + \frac{1}{Rt}\phi_{,xx} - \left(\frac{\bar{N}_{x0} w_{1,xx}}{t^2} + \frac{\bar{N}_{s0} w_{1,ss}}{t^2} \right. \\ &\qquad \left. + w_0'' \phi_{,ss} + w^{*''} \phi_{,ss} \right) \\ &\qquad + \frac{k_e w_1}{Et^3} = 0 \end{aligned} \tag{62}$$

and

$$\nabla^4 \phi - \left(\frac{1}{Rt} w_{1,xx} - w_0'' w_{1,ss} - w^{*''} w_{1,ss}\right) = 0 \tag{63}$$

Where ($''$) shows the second derivative with respect to x . Variables w_1 and ϕ are the non-dimensional displacement and stress functions, respectively. Introducing prebuckling coefficients from Eq. (61) into Eqs. (62) and (63), leads to the coupled linear stability and compatibility equations as:

$$\begin{aligned} &C\nabla^4 w_1 + \frac{1}{Rt}\phi_{,xx} - \{ \frac{p w_{1,xx}}{2\pi R E t^3} \\ &\qquad + [\frac{A}{Rt}\cos(\frac{m\pi x}{l}) + \frac{\nu P}{2\pi E R t^2 (Et + k_e R^2)} \\ &- \frac{\nu p}{2\pi R E t^3}] w_{1,ss} - \frac{m^2\pi^2}{l^2}(A-\mu)\phi_{,ss}\cos(\frac{m\pi x}{l})\} \\ &+ \frac{k_e w_1}{Et^3} = 0 \end{aligned} \tag{64}$$

and

$$\begin{aligned} &\nabla^4 \phi - [\frac{1}{Rt} w_{1,xx} + \frac{m^2\pi^2}{l^2}(A-\mu) w_{1,ss} \\ &\cos(\frac{m\pi x}{l})] = 0 \end{aligned} \tag{65}$$

Where $C = \dfrac{1}{12(1-\upsilon^2)}$

10. Application of the Galerkin method

To solve the system of Eqs. (64) and (65), considering the simply supported boundary conditions, the approximate solutions may be considered as:

$$\begin{aligned} &w_1 = B\cos(\frac{m\pi x}{l})\cos(\frac{ns}{R}), \\ &\phi = F\cos(\frac{m\pi x}{l})\cos(\frac{ns}{R}) \end{aligned} \tag{66}$$

Where m and n are the axial half-wave number and the circumferential full-wave number, respectively. B and F are coefficients that depend on m and n . Substituting the approximate solutions (66) into Eqs. (64) and (65) gives the residues R' and R'' as follows:

$$\begin{cases} R_1 + R_2 = R' \\ R_3 + R_4 = R'' \end{cases} \tag{67}$$

Where

$$\begin{aligned} &R_1 = F(\frac{m^2\pi^2}{l^2} + \frac{n^2}{R^2})^2 \cos(\frac{m\pi x}{l})\cos(\frac{ns}{R}) \\ &R_2 = B[\frac{m^2\pi^2}{Rtl^2} + (A-\mu)\frac{m^2\pi^2}{l^2}.\frac{n^2}{R^2}\cos(\frac{m\pi x}{l})] \\ &\quad \cos(\frac{m\pi x}{l})\cos(\frac{ns}{R}) \\ &R_3 = F[-\frac{m^2\pi^2}{Rtl^2} + (\mu - A)\frac{m^2\pi^2}{l^2}\frac{n^2}{R^2}\cos(\frac{m\pi x}{l})] \\ &\quad \cos(\frac{m\pi x}{l})\cos(\frac{ns}{R}) \\ &R_4 = B\left\{ C\left(\frac{m^2\pi^2}{l^2} + \frac{n^2}{R^2}\right)^2 - \frac{P}{2\pi R E t^3}.\frac{m^2\pi^2}{l^2} \right. \\ &\qquad + \frac{An^2}{R^3 t}\cos\left(\frac{m\pi x}{l}\right) \\ &\qquad \left. + \frac{\upsilon P n^2}{2\pi R^3 t^2 \left(Et + k_e R^2\right)} - \frac{\upsilon P n^2}{2\pi R^3 t^3} + \frac{k_e}{Et^3} \right\} \\ &\qquad \cos\left(\frac{m\pi x}{l}\right)\cos\left(\frac{ns}{R}\right) \end{aligned} \tag{68}$$

Application of the Galerkin method to Eq. (67) leads to:

$$\begin{aligned} &H_{11} = \frac{1}{F}\int_0^{2\pi R}\int_{-\frac{l}{2}}^{\frac{l}{2}} R_1 \cos(\frac{m\pi x}{l})\cos(\frac{ns}{R})dxds \\ &H_{12} = \frac{1}{B}\int_0^{2\pi R}\int_{-\frac{l}{2}}^{\frac{l}{2}} R_2 \cos(\frac{m\pi x}{l})\cos(\frac{ns}{R})dxds \end{aligned} \tag{69}$$

$$H_{21}=\frac{1}{F}\int_0^{2\pi R}\int_{-\frac{l}{2}}^{\frac{l}{2}} R_3\cos(\frac{m\pi x}{l})\cos(\frac{ns}{R})dxds$$
$$H_{22}=\frac{1}{B}\int_0^{2\pi R}\int_{-\frac{l}{2}}^{\frac{l}{2}} R_4\cos(\frac{m\pi x}{l})\cos(\frac{ns}{R})dxds$$

It is noted that:

$$\int_{-\frac{l}{2}}^{\frac{l}{2}}\cos^2(\frac{m\pi x}{l})dx=\frac{l}{2}$$
$$\int_{-\frac{l}{2}}^{\frac{l}{2}}\cos^3(\frac{m\pi x}{l})dx=\frac{2l}{m\pi}[\sin\frac{m\pi}{2}-\frac{1}{3}\sin^3\frac{m\pi}{2}] \quad (70)$$
$$\int_0^{2\pi R}\cos^2(\frac{ns}{R})ds=\pi R$$

Thus:

$$H_{11}=(\frac{m^2\pi^2}{l^2}+\frac{n^2}{R^2})^2\frac{\pi Rl}{2}$$
$$H_{12}=\{\frac{m^2\pi^2}{l^2Rt}\frac{\pi Rl}{2}+\left(A-\mu\right)\frac{m^2\pi^2}{l^2}\frac{n^2}{R^2}\frac{2Rl}{m}\left[\sin\left(\frac{m\pi}{2}\right)-\frac{1}{3}\sin^3\left(\frac{m\pi}{2}\right)\right]\}$$
$$H_{21}=\{-\frac{m^2\pi^2}{l^2Rt}\frac{\pi Rl}{2}-\left(\mu-A\right)\frac{m^2\pi^2}{l^2}\frac{n^2}{R^2}\frac{2Rl}{m}\left[\sin\left(\frac{m\pi}{2}\right)-\frac{1}{3}\sin^3\left(\frac{m\pi}{2}\right)\right]\} \quad (71)$$
$$H_{22}=\{\left[C\left(\frac{m^2\pi^2}{l^2}+\frac{n^2}{R^2}\right)-\frac{P}{2\pi REt^3}\frac{m^2\pi^2}{l^2}^2+\frac{\upsilon Pn^2}{2\pi R^3t^2\left(Et+k_eR^2\right)}-\frac{\upsilon Pn^2}{2\pi R^3t^3}+\frac{k_e}{Et^3}\right]\frac{\pi Rl}{2}+\frac{An^2}{R^3t}\frac{2Rl}{m}\left[\sin\left(\frac{m\pi}{2}\right)-\frac{1}{3}\sin^3\left(\frac{m\pi}{2}\right)\right]\}$$

The determinant of coefficients for even values of m has no result, but $m=4k\pm1$ (odd values of m) leads to:

$$\begin{vmatrix}H_{11} & H_{12}\\ H_{21} & H_{22}\end{vmatrix}=0\Rightarrow H_{11}H_{22}-H_{21}H_{12}=0 \quad (72)$$

This leads to the following equations in terms of P :

$$C\left(\frac{m^2\pi^2}{l^2}+\frac{n^2}{R^2}\right)^4-\frac{P}{2\pi REt^3}\frac{m^2\pi^2}{l^2}\left(\frac{m^2\pi^2}{l^2}+\frac{n^2}{R^2}\right)^2+\frac{\upsilon Pn^2}{2\pi R^3t^2\left(Et+k_eR^2\right)}\left(\frac{m^2\pi^2}{l^2}+\frac{n^2}{R^2}\right)^2-\frac{\upsilon Pn^2}{2\pi R^3Et^3}\left(\frac{m^2\pi^2}{l^2}+\frac{n^2}{R^2}\right)^2+\frac{k_e}{Et^3}\left(\frac{m^2\pi^2}{l^2}+\frac{n^2}{R^2}\right)^2$$
$$+\frac{4An^2}{R^3tm\pi}\left[\sin\left(\frac{m\pi}{2}\right)-\frac{1}{3}\sin^3\left(\frac{m\pi}{2}\right)\right].\left(\frac{m^2\pi^2}{l^2}+\frac{n^2}{R^2}\right)^2+\frac{1}{R^2t^2}\frac{m^4\pi^4}{l^4}-2\left(\mu-A\right)\frac{1}{Rt}\frac{m^4\pi^4}{l^4}\frac{n^2}{R^2}\frac{4}{m\pi}\left[\sin\left(\frac{m\pi}{2}\right)-\frac{1}{3}\sin^3\left(\frac{m\pi}{2}\right)\right]$$
$$+\left(\mu-A\right)^2\frac{m^4\pi^4}{l^4}\frac{n^4}{R^4}\frac{16}{m^2\pi^2}\left[\sin\left(\frac{m\pi}{2}\right)-\frac{1}{3}\sin^3\left(\frac{m\pi}{2}\right)\right]^2=0 \quad (73)$$

Solving Eq. (73) to find P then:

$$\frac{P}{2\pi R}=\frac{R'}{S} \quad (74)$$

Where R' and S are defined as follows:

$$R'=C\left(\frac{m^2\pi^2}{l^2}+\frac{n^2}{R^2}\right)^4+\frac{k_e}{Et^3}\left(\frac{m^2\pi^2}{l^2}+\frac{n^2}{R^2}\right)^2+\frac{4An^2}{R^3tm\pi}\left[\sin\left(\frac{m\pi}{2}\right)-\frac{1}{3}\sin^3\left(\frac{m\pi}{2}\right)\right].\left(\frac{m^2\pi^2}{l^2}+\frac{n^2}{R^2}\right)^2+\frac{1}{R^2t^2}\frac{m^4\pi^4}{l^4}$$
$$-2\left(\mu-A\right)\frac{1}{Rt}\frac{m^4\pi^4}{l^4}\frac{n^2}{R^2}\frac{4}{m\pi}\left[\sin\left(\frac{m\pi}{2}\right)-\frac{1}{3}\sin^3\left(\frac{m\pi}{2}\right)\right]+\left(\mu-A\right)^2\frac{m^4\pi^4}{l^4}\frac{n^4}{R^4}\frac{16}{m^2\pi^2}\left[\sin\left(\frac{m\pi}{2}\right)-\frac{1}{3}\sin^3\left(\frac{m\pi}{2}\right)\right]^2 \quad (75)$$

$$S=\frac{\frac{1}{Et^3}\frac{m^2\pi^2}{l^2}\left(\frac{m^2\pi^2}{l^2}+\frac{n^2}{R^2}\right)^2-\frac{\upsilon n^2}{R^2t^2\left(Et+k_eR^2\right)}}{\left(\frac{m^2\pi^2}{l^2}+\frac{n^2}{R^2}\right)^2+\frac{\upsilon n^2}{R^2Et^3}\left(\frac{m^2\pi^2}{l^2}+\frac{n^2}{R^2}\right)^2}$$

The critical axial load**,** in which buckling occurs, can be obtained from Eq. (74), where P_{cr} is the axial load, obtained by minimizing the solutions of Eq. (74) with respect to m and n.

11. Proposed prototype specifications

In this study the shell has a uniform thickness of *t=5mm*, the radius range of $0.1m \le R \le 1m$ and length $l = 1m$ with an elastic core with uniform thickness of $C = 5t$. The shell is considered to be simply supported at both ends.

The shell is considered to be from ASTM-A723 steel with a core made from aluminum 7050-T7651. The material properties are tabulated in Table 1.

12. Results and discussion

Considering the above specifications, the critical force per unit circumferential length of the cylindrical shell with an elastic core N_{xcr}, versus aspect ratio R/t is obtained from Eq. (34) and is shown in Fig. 3. It is obvious that as the shell radius increases, the buckling load decreases.

The critical axial force of the cylindrical shell with an elastic core is plotted against the ratio of length to shell radius $\frac{l}{R}$ in Fig. 4. This figure depicts that the buckling load decreases significantly by increasing $\frac{l}{R}$.

Figure 5 shows the ratio of elastic buckling load for uniaxial loading of a cylindrical shell with an elastic core to that of a cylinder without a core with an equal mass and radius, plotted against the ratio of shell radius to thickness for the shell with core. This figure shows that the ratio of the critical loads ($\frac{(P_{cr})_{with\ core}}{(P_{cr})_{eq}}$) linearly increases as the shell radius to thickness ratio ($\frac{R}{t}$) increases. It is concluded that the presence of an

Table 1. Material properties.

Property / Material	E	υ	ρ
ASTM-723	$197Gpa$	0.3	$7858\frac{kg}{m^3}$
7050-T7651	$70Gpa$	0.3	$2770\frac{kg}{m^3}$

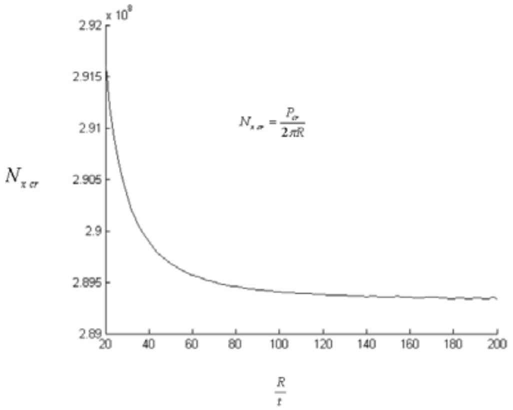


Fig. 3. Critical axial force per unit circumferential length of the cylindrical shell with a core versus aspect ratio ratio ($\frac{R}{t}$).

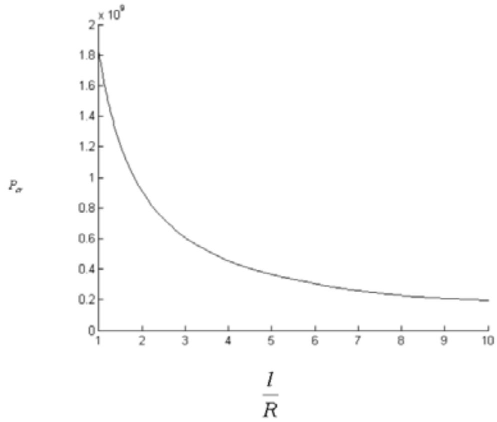


Fig. 4. Critical axial force of the cylindrical shell with a core versus aspect ratio ($\frac{l}{R}$).

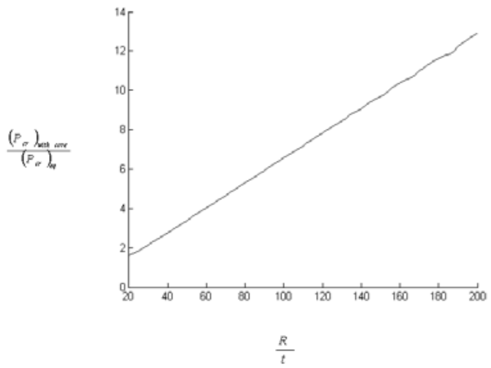


Fig. 5. Critical load ratio of the cylindrical shell with and without a core versus aspect ratio ($\frac{R}{t}$).

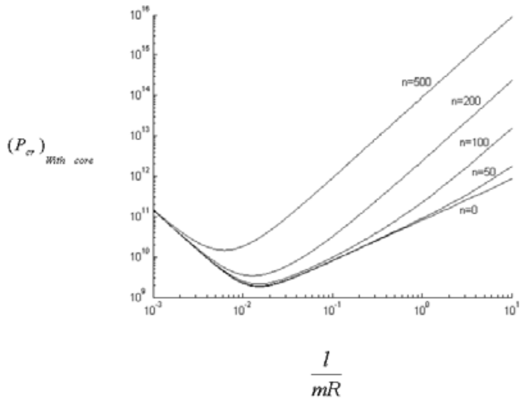
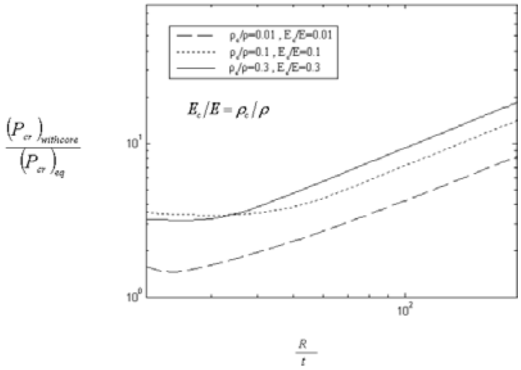
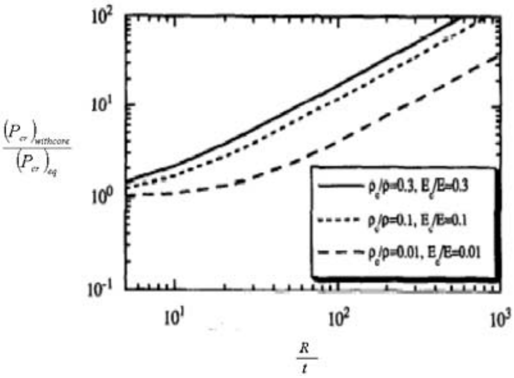


Fig. 6. The variation of axial compressive buckling load versus l/mR for different values of n.



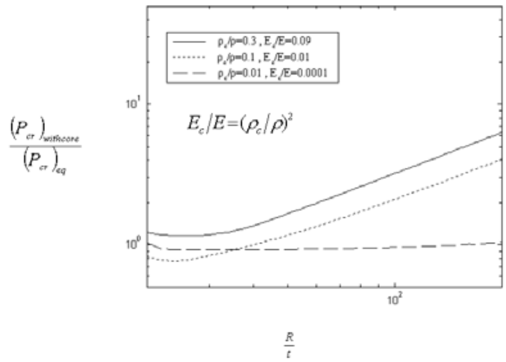
(a)



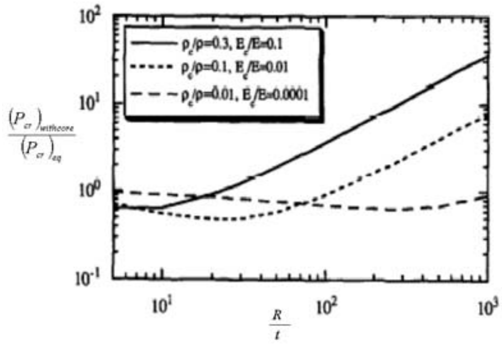
(b)

Fig. 7. (a) Critical load ratio versus aspect ratio ($\frac{R}{t}$) for $E_c/E = \rho_c/\rho$, (b) Critical load ratio versus aspect ratio ($\frac{R}{t}$) for $E_c/E = \rho_c/\rho$ [Karam et al. (1995)].

elastic core significantly increases the buckling resistance in axial compression with respect to a hollow cylinder of equal mass and radius.



(a)



(b)

Fig. 8. (a) Critical load ratio versus aspect ratio ($\frac{R}{t}$) for $E_c/E=(\rho_c/\rho)^2$, (b) Critical load ratio versus aspect ratio ($\frac{R}{t}$) for $E_c/E=(\rho_c/\rho)^2$ [Karam et al. (1995)].

Figure 6 represents the variation of axial compressive buckling load of shell with an elastic core versus l/mR for different values of n. The horizontal axis shows the value of l/mR and the vertical axis represents $(P_{cr})_{with\ core}$. The exact relationship between $(P_{cr})_{with\ core}$ and l/mR is obtained from Eq. (34) and for *n*=0, 50, 100, 200 and 500. P_{cr} against l/mR is plotted in Fig. 6, where each value of *n* corresponds to its associated mode of buckling. For each value of *n*, the critical load is minimum. Thus the critical load is the minimum point of the curve in Fig 6.

The ratio of critical buckling loads for uniaxial loading is plotted in Figs. 7 and 8. The results shown in Figs. 7(a) and 8(a) are in reasonable conformance with the Karam's et al. (1995) results shown in Figs. 7(b) and 8(b). They assumed that the ratio of the core to shell Young's module varies as the ratio of the core to shell density raised to a power of one or two, depending on the core type, i.e., honeycomb or foam core support. Supporting a shell by a core with

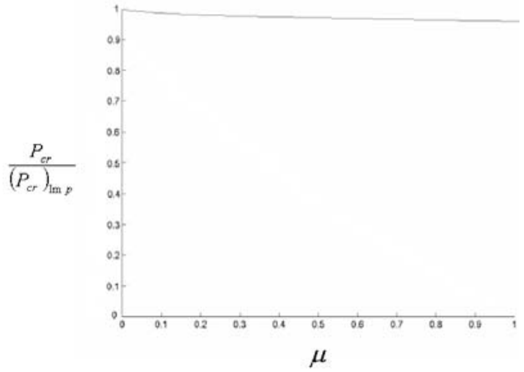


Fig. 9. Influence of imperfection magnitude on critical axial load of the cylindrical shell with a core.

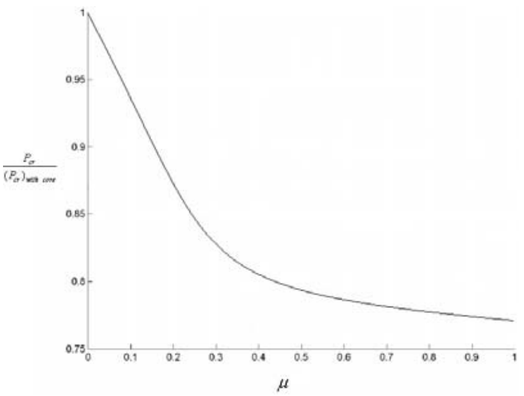


Fig. 10. Critical load ratio of the imperfect cylindrical shell without a core to that with a core versus the ratio of imperfection amplitude to shell thickness (μ).

$E_c/E = \rho_c/\rho$ leads to a substantial increase in the axial buckling load, especially at a large ratio of a/t . A core supported shell with $E_c/E = (\rho_c/\rho)^2$ shows an increase in the axial buckling load only for dense cores at high ratio of a/t .

Variation of the ratio of critical load for perfect cylindrical shell with an elastic core P_{cr} to the critical load for the corresponding imperfect shell $(P_{cr})_{\mathrm{Im}\,p}$, versus the imperfection parameter μ is plotted in Fig. 9. As the amplitude of imperfection increases, the buckling ratio is approximately constant. It is concluded that the effect of imperfection on the critical axial buckling load for a cylindrical shell with an elastic core is ignorable. Therefore, the buckling resistance of shells with a compliant core has reduced sensitivity to imperfections.

Variation of the ratio of critical load for imperfect hollow cylindrical shell P_{cr} , to the critical load for the corresponding imperfect shell with an elastic core $(P_{cr})_{with\ \ core}$, versus the imperfection parameter μ is obtained from Eq. (74) and is plotted in Fig. 10. As the amplitude of imperfection increases, the buckling ratio decreases.

13. Conclusion

A buckling analysis of a cylindrical shell with a core under axial compression loading is presented in this article. Equilibrium, stability and compatibility equations for a simply supported cylindrical shell with an elastic core are derived by the energy method. The elastic core resembles a spring with an elastic coefficient k_e , in stability and equilibrium equations. It is concluded that the buckling analysis of core filled cylindrical shells subjected to axial compression showed that cylinders with core have higher buckling loads than equivalent hollow cylinders. For a cylindrical shell with an elastic core, critical axial load curves versus the ratio of radius to thickness are shown in Figs. 3, 5, 7 and 8. These figures also include critical axial comparison curves versus the ratio of radius to thickness for cylindrical shells of equal masses with and without a core. Therefore, the application of an elastic core increases elastic stability and decreases cylindrical shell weight.

The results indicated in Figs. 7(a) and 8(a) are determined by using the energy method and have reasonable conformance with the Karam's et al. (1995) results shown in Figs. 7(b) and 8(b).

The effect of initial imperfection on the linear elastic stability of a cylindrical shell with an elastic core is analyzed by using the Galerkin method. Critical axial load curves for an imperfect shell with a core are discussed and plotted in Figs. 9 and 10 considering the ratio of deviation amplitude to shell thickness. Figure 9 shows that the buckling resistance of shells with an elastic core has less sensitivity to imperfections.

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